Correction: Série d'éxercices sur les limites

Exercice 1 *Calculons les limites :*

- $\lim\limits_{x\longrightarrow 1}\frac{3x^2-2x-1}{x-1}=\text{"}\frac{0}{0}\text{"}\quad (F.I)$

$$\lim_{x\longrightarrow 1}\frac{3x^2-2x-1}{x-1}=\lim_{x\longrightarrow 1}\frac{(3x+1)(x-1)}{x-1}=\lim_{x\longrightarrow 1}3x+1=3\times 1+1=4$$

- $\lim\limits_{x\longrightarrow 1}\frac{x^4-2x^3+x^2+x-1}{x^2+x-2}=\text{"}\frac{0}{0}\text{"}\quad (F.I)$

$$\lim_{x\longrightarrow 1}\frac{x^4-2x^3+x^2+x-1}{x^2+x-2}=\lim_{x\longrightarrow 1}\frac{(x-1)(x^3-x^2+1)}{(x+2)(x-1)}=\lim_{x\longrightarrow 1}\frac{x^3-x^2+1}{x+2}=\frac{-1+1+1}{1+2}=\frac{1}{3}$$

- $\lim\limits_{x\longrightarrow 0}\frac{\sqrt{4-x}-\sqrt{4+x}}{x}=\text{"}\frac{0}{0}\text{"}\quad (F.I)$

$$\begin{aligned}\lim_{x\longrightarrow 0}\frac{\sqrt{4-x}-\sqrt{4+x}}{x} &= \lim_{x\longrightarrow 0}\frac{\left(\sqrt{4-x}-\sqrt{4+x}\right)\left(\sqrt{4-x}+\sqrt{4+x}\right)}{x\left(\sqrt{4-x}+\sqrt{4+x}\right)}\\ &= \lim_{x\longrightarrow 0}\frac{(4-x)-(4+x)}{x\left(\sqrt{4-x}+\sqrt{4+x}\right)}\\ &= \lim_{x\longrightarrow 0}\frac{-2x}{x\left(\sqrt{4-x}+\sqrt{4+x}\right)}\\ &= \lim_{x\longrightarrow 0}\frac{-2}{\sqrt{4-x}+\sqrt{4+x}}=\frac{-2}{\sqrt{4-0}+\sqrt{4+0}}=\frac{-1}{2}\end{aligned}$$

- $\lim\limits_{x\longrightarrow 0}\frac{\sqrt{x+1}-1}{x^2-x}=\text{"}\frac{0}{0}\text{"}\quad (F.I)$

$$\begin{aligned}\lim_{x\longrightarrow 0}\frac{\sqrt{x+1}-1}{x^2-x} &= \lim_{x\longrightarrow 0}\frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}{x(x-1)\left(\sqrt{x+1}+1\right)}\\ &= \lim_{x\longrightarrow 0}\frac{(x+1)-1}{x(x-1)\left(\sqrt{x+1}+1\right)}\\ &= \lim_{x\longrightarrow 0}\frac{x}{x(x-1)\left(\sqrt{x+1}+1\right)}\\ &= \lim_{x\longrightarrow 0}\frac{1}{(x-1)\left(\sqrt{x+1}+1\right)}=\frac{1}{(0-1)\left(\sqrt{0+1}+1\right)}=-\frac{1}{2}\end{aligned}$$

Exercice 2 *Calculons les limites :*

- $\lim\limits_{x\longrightarrow 2}\frac{x^{2n}-4^n}{x^2-3x+2}=\text{"}\frac{0}{0}\text{"}\ (F.I)$

En utilisant l'identité remarquable suivante :

$$a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2}b+...+ab^{n-2}+b^{n-1}\right)=(a-b)\sum_{k=0}^{n-1}a^{n-1-k}b^k$$

Donc :

$$\begin{aligned}
\lim_{x\longrightarrow 2}\frac{x^{2n}-4^n}{x^2-3x+2} &= \lim_{x\longrightarrow 2}\frac{(x^2)^n-4^n}{(x-1)(x-2)}\\
&= \lim_{x\longrightarrow 2}\frac{(x^2-4)\sum\limits_{k=0}^{n-1}(x^2)^{n-1-k}4^k}{(x-1)(x-2)}\\
&= \lim_{x\longrightarrow 2}\frac{(x-2)(x+2)\sum\limits_{k=0}^{n-1}(x^2)^{n-1-k}4^k}{(x-1)(x-2)}\\
&= \lim_{x\longrightarrow 2}\frac{(x+2)\sum\limits_{k=0}^{n-1}(x^2)^{n-1-k}4^k}{x-1}\\
&= 4\sum_{k=0}^{n-1}4^{n-1-k}4^k=4^n\sum_{k=0}^{n-1}1=4^n(n-1+1)=4^nn
\end{aligned}$$

- $\lim\limits_{x\longrightarrow 0}\frac{\sqrt{x+1}+\sqrt{x+4}-3}{x}=\text{"}\frac{0}{0}\text{"}\ (F.I)$

$$\begin{aligned}
\lim_{x\longrightarrow 0}\frac{\sqrt{x+1}+\sqrt{x+4}-3}{x} &= \lim_{x\longrightarrow 0}\frac{\sqrt{x+1}-1+\sqrt{x+4}-2}{x}\\
&= \lim_{x\longrightarrow 0}\frac{\sqrt{x+1}-1}{x}+\frac{\sqrt{x+4}-2}{x}\\
&= \lim_{x\longrightarrow 0}\frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}{x\left(\sqrt{x+1}+1\right)}+\frac{\left(\sqrt{x+4}-2\right)\left(\sqrt{x+4}+2\right)}{x\left(\sqrt{x+4}+2\right)}\\
&= \lim_{x\longrightarrow 0}\frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)}+\frac{x+4-4}{x\sqrt{x+4}+2}\\
&= \lim_{x\longrightarrow 0}\frac{x}{x\left(\sqrt{x+1}+1\right)}+\frac{x}{x\sqrt{x+4}+2}\\
&= \lim_{x\longrightarrow 0}\frac{1}{\sqrt{x+1}+1}+\frac{1}{\sqrt{x+4}+2}\\
&= \frac{1}{\sqrt{0+1}+1}+\frac{1}{\sqrt{0+4}+2}=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}
\end{aligned}$$

- $\lim\limits_{x\longrightarrow 3}\frac{\sqrt{6+x}-3}{x^2-x-6}=" \frac{0}{0}"\ (F.I)$

$$\begin{aligned}\lim_{x\longrightarrow 3}\frac{\sqrt{6+x}-3}{x^2-x-6} &= \lim_{x\longrightarrow 3}\frac{\left(\sqrt{6+x}-3\right)\left(\sqrt{6+x}+3\right)}{(x+2)(x-3)\left(\sqrt{6+x}+3\right)}\\ &= \lim_{x\longrightarrow 3}\frac{(6+x)-9}{(x+2)(x-3)\left(\sqrt{6+x}+3\right)}\\ &= \lim_{x\longrightarrow 3}\frac{x-3}{(x+2)(x-3)\left(\sqrt{6+x}+3\right)}\\ &= \frac{1}{(3+2)\left(\sqrt{6+3}+3\right)}=\frac{1}{30}\end{aligned}$$

- $\lim\limits_{x\longrightarrow 0}\frac{1-x^2E\left(\frac{1}{x}\right)}{1+x^2E\left(\frac{1}{x}\right)}$

On sait que tout x de $\mathbb{R}^*$, on a : $\frac{1}{x}-1\prec E\left(\frac{1}{x}\right)\leq\frac{1}{x}$. *Ce qui équivaut à :*

$$\begin{aligned}x^2\left(\frac{1}{x}-1\right) &\prec x^2E\left(\frac{1}{x}\right)\leq x^2\times\frac{1}{x}\\ &\Longleftrightarrow x-x^2\prec x^2E\left(\frac{1}{x}\right)\leq x\end{aligned}$$

comme $\lim\limits_{x\longrightarrow 0}x-x^2=\lim\limits_{x\longrightarrow 0}x=0$. *Donc :* $\lim\limits_{x\longrightarrow 0}x^2E\left(\frac{1}{x}\right)=0$. *Ceci signifie que :*

$$\lim_{x\longrightarrow 0}\frac{1-x^2E\left(\frac{1}{x}\right)}{1+x^2E\left(\frac{1}{x}\right)}=1$$

Exercice 3 *Calculons les limites :*

- $\lim\limits_{x\longrightarrow +\infty}\sqrt{x+\sqrt{x}}-\sqrt{x}=" +\infty-\infty"\ \ (F.I)$

$$\begin{aligned}\lim_{x\longrightarrow +\infty}\sqrt{x+\sqrt{x}}-\sqrt{x} &= \lim_{x\longrightarrow +\infty}\frac{\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right)\left(\sqrt{x+\sqrt{x}}+\sqrt{x}\right)}{\left(\sqrt{x+\sqrt{x}}+\sqrt{x}\right)}\\ &= \lim_{x\longrightarrow +\infty}\frac{(x+\sqrt{x})-x}{\sqrt{x+\sqrt{x}}+\sqrt{x}}\\ &= \lim_{x\longrightarrow +\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x}}\\ &= \lim_{x\longrightarrow +\infty}\frac{\sqrt{x}}{\sqrt{x}\sqrt{1+\frac{\sqrt{x}}{x}}+\sqrt{x}}\\ &= \lim_{x\longrightarrow +\infty}\frac{\sqrt{x}}{\sqrt{x}\left(\sqrt{1+\frac{1}{\sqrt{x}}}+1\right)}\\ &= \lim_{x\longrightarrow +\infty}\frac{1}{\sqrt{1+\frac{1}{\sqrt{x}}}+1}=\frac{1}{2}\end{aligned}$$

- $\lim_{x\longrightarrow -\infty} 3x-1+\sqrt{9x^2+3x-2} = "-\infty+\infty" \quad (F.I)$

$$\begin{aligned}
\lim_{x\longrightarrow -\infty} 3x-1+\sqrt{9x^2+3x-2} &= \lim_{x\longrightarrow -\infty} \frac{\left(3x-1+\sqrt{9x^2+3x-2}\right)\left(3x-1-\sqrt{9x^2+3x-2}\right)}{3x-1-\sqrt{9x^2+3x-2}}\\
&= \lim_{x\longrightarrow -\infty} \frac{(3x-1)^2-(9x^2+3x-2)}{3x-1-\sqrt{9x^2+3x-2}}\\
&= \lim_{x\longrightarrow -\infty} \frac{9x^2-6x+1-9x^2-3x+2}{3x-1-\sqrt{9x^2+3x-2}}\\
&= \lim_{x\longrightarrow -\infty} \frac{-9x+3}{3x-1-\sqrt{9x^2+3x-2}}\\
&= \lim_{x\longrightarrow -\infty} \frac{x\left(-9+\frac{3}{x}\right)}{x\left(3-\frac{1}{x}+\sqrt{9+\frac{3}{x}-\frac{2}{x^2}}\right)}\\
&= \lim_{x\longrightarrow -\infty} \frac{-9+\frac{3}{x}}{3-\frac{1}{x}+\sqrt{9+\frac{3}{x}-\frac{2}{x^2}}} = \frac{-3}{2}
\end{aligned}$$

- $\lim_{x\longrightarrow +\infty} x+2-\sqrt{x^2-x+6} = "+\infty-\infty" \quad (F.I)$

$$\begin{aligned}
\lim_{x\longrightarrow +\infty} x+2-\sqrt{x^2-x+6} &= \lim_{x\longrightarrow +\infty} \frac{\left(x+2-\sqrt{x^2-x+6}\right)\left(x+2+\sqrt{x^2-x+6}\right)}{x+2+\sqrt{x^2-x+6}}\\
&= \lim_{x\longrightarrow +\infty} \frac{(x+2)^2-(x^2-x+6)}{x+2+\sqrt{x^2-x+6}}\\
&= \lim_{x\longrightarrow +\infty} \frac{x^2+4x+4-x^2+x-6}{x+2+\sqrt{x^2-x+6}}\\
&= \lim_{x\longrightarrow +\infty} \frac{5x-2}{x+2+\sqrt{x^2-x+6}}\\
&= \lim_{x\longrightarrow +\infty} \frac{x\left(5-\frac{2}{x}\right)}{x\left(1+\frac{2}{x}+\sqrt{1-\frac{1}{x}+\frac{6}{x^2}}\right)}\\
&= \lim_{x\longrightarrow +\infty} \frac{5-\frac{2}{x}}{1+\frac{2}{x}+\sqrt{1-\frac{1}{x}+\frac{6}{x^2}}}\\
&= \frac{5}{2}
\end{aligned}$$

Exercice 4 *Calculons les limites suivantes :*

- $\lim_{x\longrightarrow \frac{\pi}{4}} \frac{2\cos x-\sqrt{2}}{x-\frac{\pi}{4}} = "\frac{0}{0}" \quad (F.I)$

On pose : $x-\frac{\pi}{4}=X$, *si* x *tend vers* $\frac{\pi}{4}$, *alors* X *tend vers* 0. *On obtient* :

$$\begin{aligned}
\lim_{x\longrightarrow\frac{\pi}{4}}\frac{2\cos x-\sqrt{2}}{x-\frac{\pi}{4}} &= \lim_{X\longrightarrow 0}\frac{2\cos\left(\frac{\pi}{4}+X\right)-\sqrt{2}}{X}\\
&= \lim_{X\longrightarrow 0}\frac{2\left(\cos\frac{\pi}{4}\cos X-\sin\frac{\pi}{4}\sin X\right)-\sqrt{2}}{X}\\
&= \lim_{X\longrightarrow 0}\frac{\sqrt{2}\cos X-\sqrt{2}\sin X-\sqrt{2}}{X}\\
&= \lim_{X\longrightarrow 0}\frac{-\sqrt{2}\left(1-\cos X\right)-\sqrt{2}\sin X}{X}\\
&= \lim_{X\longrightarrow 0}-\sqrt{2}\times\frac{1-\cos X}{X^2}\times X-\sqrt{2}\times\frac{\sin X}{X}\\
&= -\sqrt{2}\times\frac{1}{2}\times 0-\sqrt{2}\times 1=-\sqrt{2}
\end{aligned}$$

- $\lim_{x\longrightarrow 1}\frac{\sin(\pi x)}{x-1}="\frac{0}{0}"\quad(F.I)$

On pose : $x-1=X$, *si* x *tend vers* 1 *alors* X *tend vers* 0. *On obtient* :

$$\begin{aligned}
\lim_{x\longrightarrow 1}\frac{\sin(\pi x)}{x-1} &= \lim_{X\longrightarrow 0}\frac{\sin\left(\pi\left(X+1\right)\right)}{X}\\
&= \lim_{X\longrightarrow 0}\frac{\sin\left(\pi X+\pi\right)}{X}\\
&= \lim_{X\longrightarrow 0}\frac{-\sin\pi X}{X}\qquad,\quad \sin\left(\pi+x\right)=-\sin x\\
&= \lim_{X\longrightarrow 0}-\frac{\sin\pi X}{\pi X}\times\pi\\
&= -\pi
\end{aligned}$$

- $\lim_{x\longrightarrow\frac{\pi}{4}}\frac{1-\sqrt{2}\cos x}{1-\sqrt{2}\sin x}="\frac{0}{0}"\ (F.I)$

$$\begin{aligned}
\lim_{x\longrightarrow\frac{\pi}{4}}\frac{1-\sqrt{2}\cos x}{1-\sqrt{2}\sin x} &= \lim_{x\longrightarrow\frac{\pi}{4}}\frac{\sqrt{2}\left(\frac{\sqrt{2}}{2}-\cos x\right)}{\sqrt{2}\left(\frac{\sqrt{2}}{2}-\sin x\right)}\\
&= \lim_{x\longrightarrow\frac{\pi}{4}}\frac{\cos\frac{\pi}{4}-\cos x}{\sin\frac{\pi}{4}-\sin x}\\
&= \lim_{x\longrightarrow\frac{\pi}{4}}\frac{-2\sin\left(\frac{\frac{\pi}{4}+x}{2}\right)\sin\left(\frac{\frac{\pi}{4}-x}{2}\right)}{2\sin\left(\frac{\frac{\pi}{4}-x}{2}\right)\cos\left(\frac{\frac{\pi}{4}+x}{2}\right)}\\
&= \lim_{x\longrightarrow\frac{\pi}{4}}-\frac{\sin\left(\frac{\pi}{8}+\frac{x}{2}\right)}{\cos\left(\frac{\pi}{8}+\frac{x}{2}\right)}\\
&= \lim_{x\longrightarrow\frac{\pi}{4}}-\tan\left(\frac{\pi}{8}+\frac{x}{2}\right)=-\tan\left(\frac{\pi}{8}+\frac{\pi}{8}\right)=-\tan\frac{\pi}{4}=-1
\end{aligned}$$

- $\lim_{x\longrightarrow 0} \frac{x\cos x+\sin x}{\sqrt{x+1}-1} = "\frac{0}{0}" \ (F.I)$

$$\begin{aligned}\lim_{x\longrightarrow 0} \frac{x\cos x+\sin x}{\sqrt{x+1}-1} &= \lim_{x\longrightarrow 0} \frac{x\left(\cos x+\frac{\sin x}{x}\right)\left(\sqrt{x+1}+1\right)}{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}\\ &= \lim_{x\longrightarrow 0} \frac{x\left(\cos x+\frac{\sin x}{x}\right)\left(\sqrt{x+1}+1\right)}{x}\\ &= \lim_{x\longrightarrow 0} \left(\cos x+\frac{\sin x}{x}\right)\left(\sqrt{x+1}+1\right) = 4\end{aligned}$$

Exercice 5 *Soit $n \in \mathbb{N}^*$, on considère la fonction numérique f_n définie par :*

$$f_n(x) = \frac{n-\sin x-\sin^2 x-\sin^3 x-...-\sin^n x}{1-\sin^2 x}$$

1. *On cherche D_f :*

$$D_f = \left\{x\in\mathbb{R}/\ 1-\sin^2 x\neq 0\right\}$$

On résout dans $\mathbb{R}$ l'équation suivante : $1-\sin^2 x = 0$

$$\begin{aligned}1-\sin^2 x &= 0 \iff (1+\sin x)(1-\sin x)=0\\ &\iff \sin x = 1 \quad ou \quad \sin x = -1\\ &\iff x=\frac{\pi}{2}+2k\pi \ \ ou\ x=\frac{-\pi}{2}+2k\pi \quad / \ k\in\mathbb{Z}\end{aligned}$$

Donc :

$$\begin{aligned}D_f &= \left\{x\in\mathbb{R}/\ x\neq\frac{\pi}{2}+2k\pi \quad et \quad x\neq\frac{-\pi}{2}+2k\pi\ /\ k\in\mathbb{Z}\right\}\\ &= \mathbb{R}\backslash\left\{\frac{\pi}{2}+2k\pi,\frac{-\pi}{2}+2k\pi/\ k\in\mathbb{Z}\right\}\end{aligned}$$

2. *Monrons que :* $\lim_{x\longrightarrow\frac{\pi}{2}} \frac{1-\sin^k x}{1-\sin^2 x}$

$$\begin{aligned}\lim_{x\longrightarrow\frac{\pi}{2}} \frac{1-\sin^k x}{1-\sin^2 x} &= \lim_{x\longrightarrow\frac{\pi}{2}} \frac{(1-\sin x)\sum\limits_{i=0}^{k-1} 1^{k-1-i}(\sin x)^i}{(1-\sin x)(1+\sin x)}\\ &= \lim_{x\longrightarrow\frac{\pi}{2}} \frac{\sum\limits_{i=0}^{k-1}(\sin x)^i}{(1+\sin x)}\\ &= \frac{\sum\limits_{i=0}^{k-1} 1^i}{2} = \frac{k-1-0+1}{2} = \frac{k}{2}\end{aligned}$$

3.

$$
\begin{aligned}
\lim_{x\longrightarrow\frac{\pi}{2}} f_n(x) &= \lim_{x\longrightarrow\frac{\pi}{2}} \frac{n-\sin x-\sin^2 x-\sin^3 x-...-\sin^n x}{1-\sin^2 x} \\
&= \lim_{x\longrightarrow\frac{\pi}{2}} \frac{n-\sum\limits_{k=1}^{n}\sin^k x}{1-\sin^2 x} \\
&= \lim_{x\longrightarrow\frac{\pi}{2}} \frac{n+\sum\limits_{k=1}^{n}\left(1-\sin^k x-1\right)}{1-\sin^2 x} \\
&= \lim_{x\longrightarrow\frac{\pi}{2}} \frac{n+\sum\limits_{k=1}^{n}1-\sin^k x-\sum\limits_{k=1}^{n}1}{1-\sin^2 x} \\
&= \lim_{x\longrightarrow\frac{\pi}{2}} \frac{\sum\limits_{k=1}^{n}1-\sin^k x}{1-\sin^2 x} \\
&= \lim_{x\longrightarrow\frac{\pi}{2}} \sum_{k=1}^{n}\left(\frac{1-\sin^k x}{1-\sin^2 x}\right) \\
&= \frac{1}{2}\sum_{k=1}^{n}k=\frac{n\left(n+1\right)}{4}
\end{aligned}
$$